

Analytical Approach to Polarization Mode Dispersion in Linearly Spun Fiber With Birefringence

by Vinod K. Mishra

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Computational and Information Sciences Directorate, ARL

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1. Introduction

In 2003, Wang et al.¹ derived expressions for the Differential Group Delay (DGD) of a randomly birefringent fiber in the Fixed Modulus Model (FMM). The DGD has both modulus and phase. The FMM assumes that the modulus of the birefringence vector is a random variable. Wang et al. presented analytical results with the following assumptions:

- The spin function is periodic.
- The periodicity length (p) of the fiber is much smaller than the fiber correlation length (L_F) or $p \ll L_F$.

Later they also generalized the FMM and presented the Random Modulus Model (RMM), which also includes the randomness in the direction of the birefringence vector. Now the RMM equations could only be solved numerically.

In the present work, the full implications of the FMM have been explored under the following conditions:

- The $p \ll L_F$ approximation has been relaxed.
- A nonzero twist has been included.
- A constant spin rate has been added.

We give the analytical solutions of the exact FMM equations and present some numerical results showing the effect of different physical conditions.

2. Theoretical Analysis

2.1 The Model With Periodic Spin Function

The starting point is the well-known vector equation describing the change in the Jones local electric field vector $\vec{A}(\omega, z)$ with the angular frequency ω and distance z along a twisted fiber.

$$\begin{bmatrix} dA_1(z)/dz \\ dA_2(z)/dz \end{bmatrix} = \frac{i}{2}(\Delta\beta) \begin{bmatrix} 0 & e^{2i\Theta(z)} \\ e^{-2i\Theta(z)} & 0 \end{bmatrix} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} \quad (1)$$

¹ Wang, M.; Li, T.; Jian, S. Analytical Theory for Polarization Mode Dispersion of Spun and Twisted Fiber. *Optics Express* **2003**, *11* (19), 2403–2410.

Here $\Delta\beta$ (ω) is the birefringence and

$$\Theta(z) = \frac{\alpha_0}{\eta} \sin(\eta z) \quad (2)$$

is the periodic spin profile function with spin magnitude α_0 and angular frequency of spatial modulation η .

The boundary conditions are

$$A_1(0) = 1, dA_1(0)/dz = 0 \quad (3a)$$

$$A_2(0) = 0, dA_2(0)/dz = i(\Delta\beta/2) \quad (3b)$$

Let $s = \eta z$ be a dimensionless variable. We use $(d/dz) = \eta(d/ds)$ to rewrite equation 1.

$$\begin{bmatrix} A_{1s}(s) \\ A_{2s}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2icsin s} \\ e^{-2icsin s} & 0 \end{bmatrix} \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \quad (4)$$

The subscripts denote differentiation ($A_{1s} = dA_1/ds, A_{2s} = dA_2/ds$). Also, $a = (\Delta\beta/2\eta)$ and $c = (\alpha_0/\eta)$ are dimensionless constants.

We express all parameters in terms of the lengths given as beat length ($L_B = 2\pi/\Delta\beta$), spin period ($\Lambda = 2\pi/\eta$), and coupling length ($l_0 = 2\pi/\alpha_0$). Then we can write $a = \Lambda/2L_B, c = L_B/l_0$.

The new boundary conditions are

$$A_1(0) = 1, A_{1s}(0) = 0 \quad (5a)$$

$$A_2(0) = 0, A_{2s}(0) = ia \quad (5b)$$

These equations (equation 1 or, equivalently, equation 4) do not have analytical solutions.

In the perturbative approximation (see appendix), an analytical result has been derived earlier.¹ In the present work, we derive analytic solutions by approximating the sine function by linear segments and compare them to the perturbative solutions for the same segments.

2.2 Linear Segment Approximation to the Periodic Spin Function: Analytical Solutions for the Jones Amplitude Equations

2.2.1 The Model

The periods of the straight line segments shown in figure 1 approximate the periodic sine function. A single period with three-segment approximation is shown.

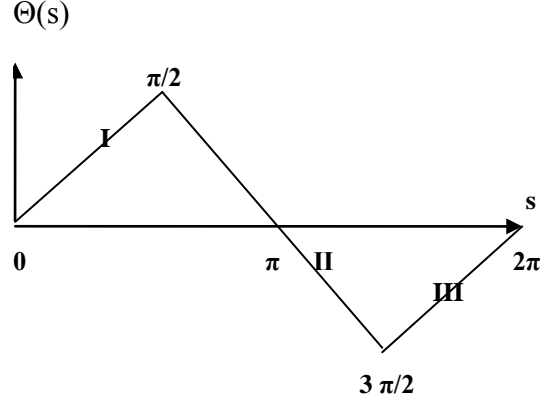


Figure 1. The three-segment approximation to the periodic sine function.

The field amplitudes for a given segment satisfy the following equations.

$$\begin{bmatrix} A_{1s}^{(m)}(s) \\ A_{2s}^{(m)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_m(s)} \\ e^{-2i\theta_m(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(m)}(s) \\ A_2^{(m)}(s) \end{bmatrix} \quad (6)$$

The superscript and subscript m both indicate the segments for which the coupled equations hold. The limits of segments follow.

We require that the endpoints of $\theta_m(s)$ should be the same as that of the sine function spin profile $\Theta(s)|_{spin} = c \sin s$ for all segments. Define $\tilde{c} = (2c/\pi)$ so that the end-point conditions for segments hold.

For $n=1$, Segment I: $0 \leq s \leq \pi/2$,

$$\begin{aligned} \theta_1(s) &= \tilde{c}s \\ \Theta(s=0)|_{spin} &= 0 = \theta_1(s=0), \\ \Theta(s=\pi/2)|_{spin} &= c = \theta_1(s=\pi/2) \end{aligned}$$

For $n=2$, Segment II: $\pi/2 \leq s \leq 3\pi/2$,

$$\begin{aligned} \theta_2(s) &= -\tilde{c}s + 2c \\ \Theta(s=\pi/2)|_{spin} &= c = \theta_2(s=\pi/2), \\ \Theta(s=3\pi/2)|_{spin} &= -c = \theta_2(s=3\pi/2) \end{aligned}$$

For $n=3$, Segment III: $3\pi/2 \leq s \leq 2\pi$,

$$\theta_3(s) = \tilde{c}s - 4c$$

$$\Theta(s = 3\pi/2)|_{spin} = -c = \theta_3(s = 3\pi/2),$$

$$\Theta(s = 5\pi/2)|_{spin} = c = \theta_3(s = 5\pi/2)$$

2.2.2 The General M -Segment Solutions

The solutions for the m th segment have the following general form:

$$\begin{bmatrix} e^{-i\theta_m(s)} A_1^{(m)}(s) \\ iae^{i\theta_m(s)} A_2^{(m)}(s) \end{bmatrix} = \begin{bmatrix} a_1^{(m)} + ib_1^{(m)} & a_2^{(m)} + ib_2^{(m)} \\ \left\{ \begin{array}{l} -\theta_{m/s} b_1^{(m)} + q_m a_2^{(m)} \\ + i(\theta_{m/s} a_1^{(m)} + q_m b_2^{(m)}) \end{array} \right\} & \left\{ \begin{array}{l} -(\theta_{m/s} b_2^{(m)} + q_m a_1^{(m)}) \\ + i(\theta_{m/s} a_2^{(m)} - q_m b_1^{(m)}) \end{array} \right\} \end{bmatrix} \begin{bmatrix} \cos q_m s \\ \sin q_m s \end{bmatrix} \quad (7)$$

with

$$q_m^2 = a^2 + \theta_m^2(s), \theta_{m/s} = d\theta_m(s)/ds$$

The exact solutions for the coupled equations in one segment are related to those in the previous adjacent segment by the following chain-relations among the coefficients.

Define $u = (q_{m-1}/q_m)$, $v = (\theta_{m/s} - \theta_{m-1/s})/q_m$, and then the chain-relations are given by

$$\begin{bmatrix} a_1^{(m)} \\ a_2^{(m)} \\ b_1^{(m)} \\ b_2^{(m)} \end{bmatrix} = \left\{ \begin{bmatrix} t_1 & t_3 & 0 & 0 \\ t_2 & t_4 & 0 & 0 \\ 0 & 0 & t_1 & t_3 \\ 0 & 0 & t_2 & t_4 \end{bmatrix} + u \begin{bmatrix} t_4 & -t_2 & 0 & 0 \\ -t_3 & t_1 & 0 & 0 \\ 0 & 0 & t_4 & -t_2 \\ 0 & 0 & -t_3 & t_1 \end{bmatrix} \right\} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix} + v \begin{bmatrix} 0 & 0 & -t_2 & -t_4 \\ 0 & 0 & t_1 & t_3 \\ t_2 & t_4 & 0 & 0 \\ -t_1 & -t_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix} \quad (8)$$

Here, the matrix elements are

$$\begin{aligned} t_1 &= \cos q_{m-1} s_{m-1} \cos q_m s_{m-1}, t_2 = \cos q_{m-1} s_{m-1} \sin q_m s_{m-1}, \\ t_3 &= \sin q_{m-1} s_{m-1} \cos q_m s_{m-1}, t_4 = \sin q_{m-1} s_{m-1} \sin q_m s_{m-1}. \end{aligned}$$

The matrix chain-relations can be written compactly by expressing the 4×4 matrices as outer products (denoted by the symbol \otimes) of two 2×2 matrices as

$$\begin{bmatrix} a_1^{(m)} \\ a_2^{(m)} \\ b_1^{(m)} \\ b_2^{(m)} \end{bmatrix} = \left\{ \left(\begin{bmatrix} t_1 & t_3 \\ t_2 & t_4 \end{bmatrix} + u \begin{bmatrix} t_4 & -t_2 \\ -t_3 & t_1 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix} + v \begin{bmatrix} t_2 & t_4 \\ -t_1 & -t_3 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix} \quad (9)$$

2.2.3 The Specific 3-Segment Solutions

The details about solutions for 3-segments follow.

- Segment I: $0 \leq s \leq \pi/2$

The equations are

$$\begin{bmatrix} A_{1s}^{(1)}(s) \\ A_{2s}^{(1)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_1(s)} \\ e^{-2i\theta_1(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(1)}(s) \\ A_2^{(1)}(s) \end{bmatrix} \quad (10)$$

The boundary conditions are

$$[A_1^{(1)}(s=0)] = 1, [A_{1s}^{(1)}(s=0)] = 0 \quad (11a)$$

$$[A_2^{(1)}(s=0)] = 0, [A_{2s}^{(1)}(s=0)] = ia \quad (11b)$$

Let

$$n = (\tilde{c}/q) = \left[1 + \left(\frac{\pi d_0}{4L_B} \right) \right]^{-1/2} \quad (12)$$

then the analytical solutions are similar to those given in section 2.2

$$\begin{bmatrix} e^{-i\tilde{c}s} A_1^{(1)}(s) \\ (q/a) e^{i\tilde{c}s} A_2^{(1)}(s) \end{bmatrix} = \begin{bmatrix} 1 & -in \\ 0 & i \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \quad (13)$$

Comparison with general expression gives the following coefficients:

$$a_1^{(1)} = 1, b_1^{(1)} = 0, a_2^{(1)} = 0, b_2^{(1)} = -n \quad (14)$$

For calculating Polarization Mode Dispersion (PMD) Correction Factor (PCF), the amplitudes have to be differentiated with respect to ω , which will be denoted by subscript ω . Some useful relations needed for this are

$$\frac{d}{d\omega} \left(\frac{a}{q} \right) = n^2 \left(\frac{a_\omega}{q} \right), a_\omega = \frac{da}{d\omega} = \frac{\gamma}{2\eta}, \gamma = \frac{d(\Delta\beta)}{d\omega}, n_\omega = -n \left(\frac{a}{q} \right) \left(\frac{a_\omega}{q} \right), q_\omega = a \left(\frac{a_\omega}{q} \right) \quad (15)$$

Then we can write

$$\begin{bmatrix} (q/a)e^{-i\tilde{c}s} A_{1\omega}^{(1)}(s) \\ e^{i\tilde{c}s} A_{2\omega}^{(1)}(s) \end{bmatrix} = \begin{pmatrix} \frac{a_\omega}{q} \\ q \end{pmatrix} \begin{bmatrix} p_1^{(1)} + ip_2^{(1)} & p_3^{(1)} + ip_4^{(1)} \\ p_5^{(1)} + ip_6^{(1)} & p_7^{(1)} + ip_8^{(1)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \quad (16)$$

where

$$p_1^{(1)} = 0, p_2^{(1)} = -nqs, p_3^{(1)} = -qs, p_4^{(1)} = n \quad (17a)$$

$$p_5^{(1)} = 0, p_6^{(1)} = (1-n^2)qs, p_7^{(1)} = 0, p_8^{(1)} = n^2 \quad (17b)$$

Some interesting relations are found as

$$\Delta\beta = (4\pi q/\Lambda)\sqrt{1-n^2}, z = (\Lambda/2\pi)s, \alpha_0 = (2\pi^2 q^2/\Lambda)n\sqrt{1-n^2} \quad (18)$$

- Segment II: $\pi/2 \leq s \leq 3\pi/2$

The equations are

$$\begin{bmatrix} A_{1s}^{(2)}(s) \\ A_{2s}^{(2)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_2(s)} \\ e^{-2i\theta_2(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(2)}(s) \\ A_2^{(2)}(s) \end{bmatrix} \quad (19)$$

The boundary conditions are

$$[A_1^{(1)}(s = \pi/2)] = [A_1^{(2)}(s = \pi/2)] \quad (20a)$$

$$[A_{1s}^{(1)}(s = \pi/2)] = [A_{1s}^{(2)}(s = \pi/2)] \quad (20b)$$

Similar expressions exist for $A_2^{(2)}(s)$. Using the chain-relations with $n = 2$, the analytical solutions are obtained.

$$\begin{bmatrix} e^{-i(-\tilde{c}s+2c)} A_1^{(2)}(s) \\ (q/a)e^{i(-\tilde{c}s+2c)} A_2^{(2)}(s) \end{bmatrix} = \begin{bmatrix} 1-n^2+n^2\cos\pi q - in\sin\pi q & n(n\sin\pi q + i\cos\pi q) \\ -n(1-\cos\pi q) & n\sin\pi q + i \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \quad (21)$$

The ω -differentiated amplitudes are found as

$$\begin{bmatrix} (q/a)e^{-i(\tilde{c}s-2c)} A_{1\omega}^{(2)}(s) \\ e^{i(\tilde{c}s-2c)} A_{2\omega}^{(2)}(s) \end{bmatrix} = \begin{pmatrix} \frac{a_\omega}{q} \\ q \end{pmatrix} \begin{bmatrix} p_1^{(2)} + ip_2^{(2)} & p_3^{(2)} + ip_4^{(2)} \\ p_5^{(2)} + ip_6^{(2)} & p_7^{(2)} + ip_8^{(2)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \quad (22)$$

where

$$p_1^{(2)} = n^2 \{2(1-\cos\pi q) - \pi q \sin\pi q + qs \sin\pi q\} \quad (23a)$$

$$p_2^{(2)} = n \{\sin\pi q - \pi q \cos\pi q + qs \cos\pi q\} \quad (23b)$$

$$p_3^{(2)} = n^2 (-2\sin\pi q + \pi q \cos\pi q) - (1-n^2+n^2\cos\pi q)qs \quad (23c)$$

$$p_4^{(2)} = n \{-(\cos\pi q + \pi q \sin\pi q) + qs \sin\pi q\} \quad (23d)$$

$$p_5^{(2)} = n\{(1-2n^2)(1-\cos \pi q) - (1-n^2)\pi q \sin \pi q + (1-n^2)qs \sin \pi q\} \quad (23e)$$

$$p_6^{(2)} = (1-n^2)qs \quad (23f)$$

$$p_7^{(2)} = n\{-(1-2n^2)\sin \pi q + (1-n^2)\pi q \cos \pi q + (1-n^2)(1-\cos \pi q)qs\} \quad (23g)$$

$$p_8^{(2)} = n^2 \quad (23h)$$

- Segment III: $3\pi/2 \leq s \leq 2\pi$

The equations are

$$\begin{bmatrix} A_{1s}^{(3)}(s) \\ A_{2s}^{(3)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_3(s)} \\ e^{-2i\theta_3(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(3)}(s) \\ A_2^{(3)}(s) \end{bmatrix} \quad (24)$$

The boundary conditions are

$$[A_1^{(2)}(s=3\pi/2)] = [A_1^{(3)}(s=3\pi/2)] \quad (25a)$$

$$[A_{1s}^{(2)}(s=3\pi/2)] = [A_{1s}^{(3)}(s=3\pi/2)] \quad (25b)$$

Similar expressions exist for $A_2^{(3)}(s)$. Using the chain-relations with $n=3$, the analytical solutions are obtained.

$$\begin{aligned} & \begin{bmatrix} e^{-i(\tilde{c}s-4c)} A_1^{(3)}(s) \\ (q/a) e^{i(\tilde{c}s-4c)} A_2^{(3)}(s) \end{bmatrix} \\ &= \begin{bmatrix} 1-n^2+n^2 \cos \pi q & n^2 \sin 2\pi q \\ +in \left\{ \begin{matrix} n^2 \sin 2\pi q \\ + (1-n^2)(\sin 3\pi q - \sin \pi q) \end{matrix} \right\} & -in \left\{ \begin{matrix} n^2 \cos 2\pi q \\ + (1-n^2)(1 + \cos 3\pi q - \cos \pi q) \end{matrix} \right\} \\ \left[\begin{matrix} n(\cos \pi q - \cos 3\pi q) \\ + in^2(\sin 3\pi q - \sin 2\pi q - \sin \pi q) \end{matrix} \right] & \left[\begin{matrix} n(\sin \pi q - \sin 3\pi q) \\ + i \{ 1-n^2+n^2(\cos \pi q + \cos 2\pi q - \cos 3\pi q) \} \end{matrix} \right] \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \end{aligned} \quad (26)$$

The ω -differentiated amplitudes are found as

$$\begin{bmatrix} (q/a) e^{-i(\tilde{c}s-4c)} A_{1\omega}^{(3)}(s) \\ e^{i(\tilde{c}s-4c)} A_{2\omega}^{(3)}(s) \end{bmatrix} = \begin{pmatrix} \frac{a_\omega}{q} \\ q \end{pmatrix} \begin{bmatrix} p_1^{(3)} + ip_2^{(3)} & p_3^{(3)} + ip_4^{(3)} \\ p_5^{(3)} + ip_6^{(3)} & p_7^{(3)} + ip_8^{(3)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix} \quad (27)$$

where

$$p_1^{(3)} = 2n^2(1 - \cos 2\pi q - \pi q \sin 2\pi q) + n^2 qs \sin 2\pi q \quad (28a)$$

$$p_2^{(3)} = n \left[\begin{aligned} & -3n^2 \sin 2\pi q - (1-3n^2)(\sin 3\pi q - \sin \pi q) \\ & + \pi q \{ 2n^2 \cos 2\pi q + (1-n^2)(3\cos 3\pi q - \cos \pi q) \} \\ & - \{ n^2 \cos 2\pi q + (1-n^2)(1 - \cos \pi q + \cos 3\pi q) \} qs \end{aligned} \right] \quad (28b)$$

$$p_3^{(3)} = -2n^2(\sin 2\pi q - \pi q \cos 2\pi q) - (1 - n^2 + n^2 \cos 2\pi q)qs \quad (28c)$$

$$p_4^{(3)} = n \left[\begin{aligned} &3n^2 \cos 2\pi q + (1 - 3n^2)(1 - \cos \pi q + \cos 3\pi q) \\ &+ \pi q \{2n^2 \sin 2\pi q + (1 - n^2)(3 \sin 3\pi q - \sin \pi q)\} \\ &- \{n^2 \sin 2\pi q + (1 - n^2)(\sin 3\pi q - \sin \pi q)\}qs \end{aligned} \right] \quad (28d)$$

$$p_5^{(3)} = n(1 - 2n^2)(\cos 3\pi q - \cos \pi q) + n(1 - n^2)(\sin 3\pi q - \sin \pi q)\pi q \\ + n(1 - n^2)(\sin \pi q - \sin 3\pi q)qs \quad (28e)$$

$$p_6^{(3)} = (1 - n^2)qs + n^2 \left[\begin{aligned} &(2 - 3n^2)(\sin \pi q + \sin 2\pi q - \sin 3\pi q) \\ &+ (1 - n^2)(3 \cos 3\pi q - 2 \cos 2\pi q - \cos \pi q)\pi q \\ &- (1 - n^2)(1 - \cos \pi q - \cos 2\pi q + \cos 3\pi q)qs \end{aligned} \right] \quad (28f)$$

$$p_7^{(3)} = n(1 - 2n^2)(\sin 3\pi q - \sin \pi q) + n(1 - n^2)(\cos \pi q - 3 \cos 3\pi q)\pi q \\ + n(1 - n^2)(\cos 3\pi q - \cos \pi q)qs \quad (28g)$$

$$p_8^{(3)} = n^2 + n^2 \left[\begin{aligned} &(2 - 3n^2)(1 - \cos \pi q - \cos 2\pi q + \cos 3\pi q) \\ &+ (1 - n^2)(3 \sin 3\pi q - 2 \sin 2\pi q - \sin \pi q)\pi q \\ &+ (1 - n^2)(\sin \pi q + \sin 2\pi q - \sin 3\pi q)qs \end{aligned} \right] \quad (28h)$$

2.3 Calculation of PCF

The sum of squares of the ω -differentiated amplitudes is similar to power and can be calculated by the following expression:

$$\frac{|A_{1\omega}^{(m)}(s)|^2 + |A_{2\omega}^{(m)}(s)|^2}{(a_\omega / q)^2} \\ = (\frac{1}{2}) \left[\begin{aligned} &(1 - n^2) \{ (p_1^{(m)})^2 + (p_2^{(m)})^2 + (p_3^{(m)})^2 + (p_4^{(m)})^2 \} \\ &+ (p_5^{(m)})^2 + (p_6^{(m)})^2 + (p_7^{(m)})^2 + (p_8^{(m)})^2 \end{aligned} \right] \\ + (\frac{1}{2}) \left[\begin{aligned} &(1 - n^2) \{ (p_1^{(m)})^2 + (p_2^{(m)})^2 - (p_3^{(m)})^2 - (p_4^{(m)})^2 \} \\ &+ (p_5^{(m)})^2 + (p_6^{(m)})^2 - (p_7^{(m)})^2 - (p_8^{(m)})^2 \end{aligned} \right] \cos 2qs \\ + [(1 - n^2) \{ p_1^{(m)} p_3^{(m)} + p_2^{(m)} p_4^{(m)} \} + p_5^{(m)} p_7^{(m)} + p_6^{(m)} p_8^{(m)}] \sin 2qs \quad (29)$$

Here, m ($=1, 2, 3$) refers to segments in sequential manner.

For calculating the normalized PCF, we need a similar expression for unspun fiber:

$$\frac{[|A_{1\omega}(s)|^2 + |A_{2\omega}(s)|^2]_{unspun\ fiber}}{(a_\omega / q)^2} = (qs)^2 \quad (30)$$

Then the expression for the PCF becomes

$$PCF^{(m)}(s) = \left[\frac{|A_{1\omega}^{(m)}(s)|^2 + |A_{2\omega}^{(m)}(s)|^2}{\left[|A_{1\omega}(s)|^2 + |A_{2\omega}(s)|^2 \right]_{unspun\text{-}fiber}} \right]^{1/2} \quad (31)$$

The left-hand side of equation 31 is a function of parameters n and q and argument s . In general, the expressions are quite complicated, but for the first segment the PCF is easily calculated and is given by

$$PCF^{(1)}(s) = \sqrt{1 - n^2 \left\{ 1 - \left(\frac{\sin qs}{qs} \right)^2 \right\}} \quad (32)$$

3. Numerical Results

The physical constants $(\Delta\beta, \alpha_0, \eta)$, or equivalently (L_B, l_0, Λ) , and the parameters (n, q) appearing in the PCF expressions are related by

$$q = \left(\frac{2\Lambda}{\pi d_0} \right) \left[1 + \left(\frac{\pi d_0}{4L_B} \right) \right]^{1/2}, \quad n = \left[1 + \left(\frac{\pi d_0}{4L_B} \right) \right]^{-1/2} \quad (33)$$

We show results for sets of parameters in two limits to emphasize the difference between the exact and perturbative calculations.

3.1 The Small-Q Limit ($\Lambda < L_B$)

As shown in table 1 and figures 2 and 3, in this limit, two sets of parameters were chosen to get small q -values (less than 1). This corresponds to beat length being larger than spin period.

Table 1. PCF vs. z plots with small- q -limit parameters.

Parameters: Λ, L_B, l_0 (m)	Values (n, q)	Comments
(1,12,1)	(0.9978, 0.6379)	$\Lambda \ll L_B$
(1,5,1)	(0.9879, 0.6444)	$\Lambda < L_B$

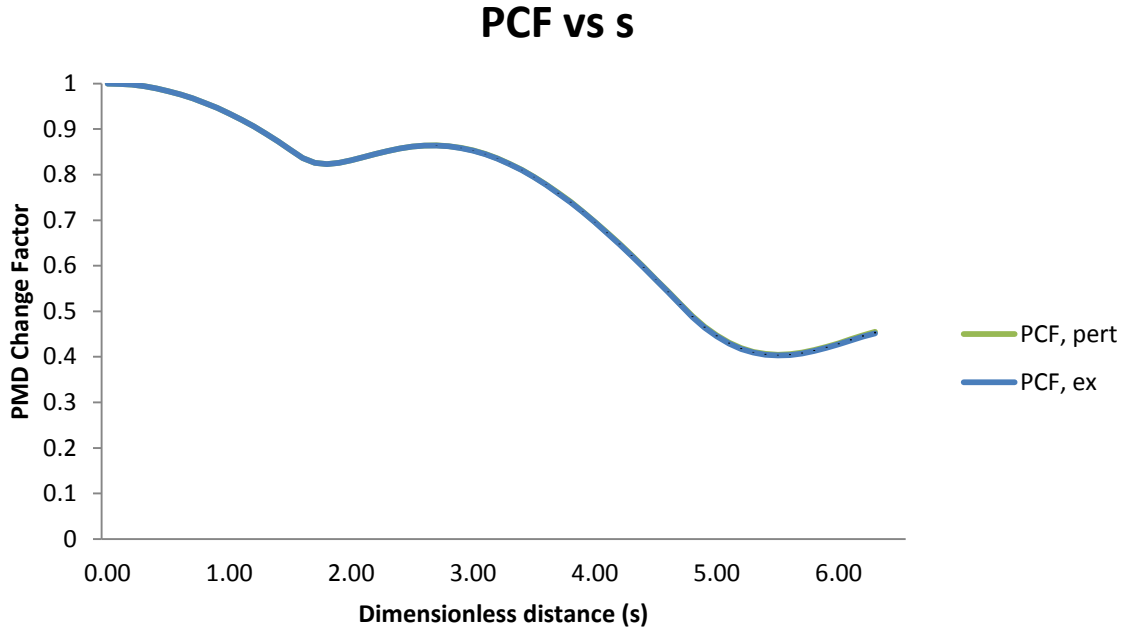


Figure 2. The PCF curve for a perturbative limit with $\Lambda = 1$ and $L_B = 12$. The curves for exact and perturbative calculations are almost identical.

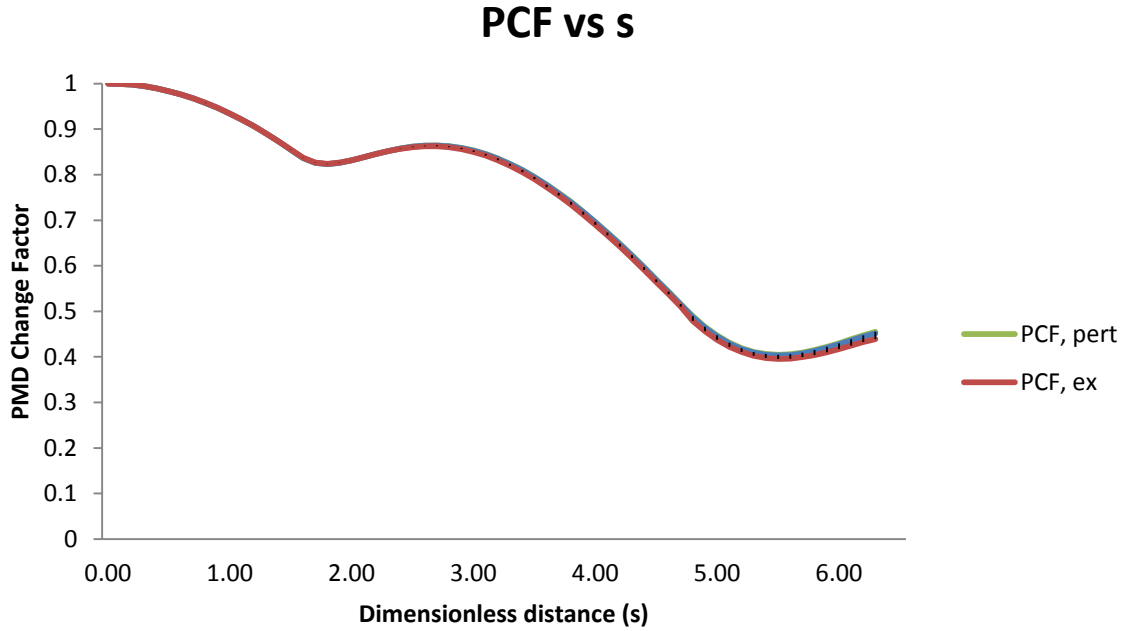


Figure 3. The PCF curve for a perturbative limit with $\Lambda = 1$ and $L_B = 5$. The curves for exact and perturbative calculations are almost identical. Note that after $s = 5$, the two start diverging a little.

3.2 The Large-Q Limit ($\Lambda > L_B$)

As shown in table 2 and figures 4 and 5, in this limit, two sets of parameters were chosen to get large q -values (much larger than 1). This corresponds to beat length being smaller than spin period.

Table 2. PCF vs. z plots with large- q -limit parameters.

Parameters: Λ, L_B, l_0 (m)	Values (n, q)	Comments
(5,1,1)	(0.7864, 4.0475)	$\Lambda > L_B$ (physical nonperturbative limit)
(12, 1, 1)	(0.7864, 9.7139)	$\Lambda \gg L_B$ (physical very nonperturbative limit)

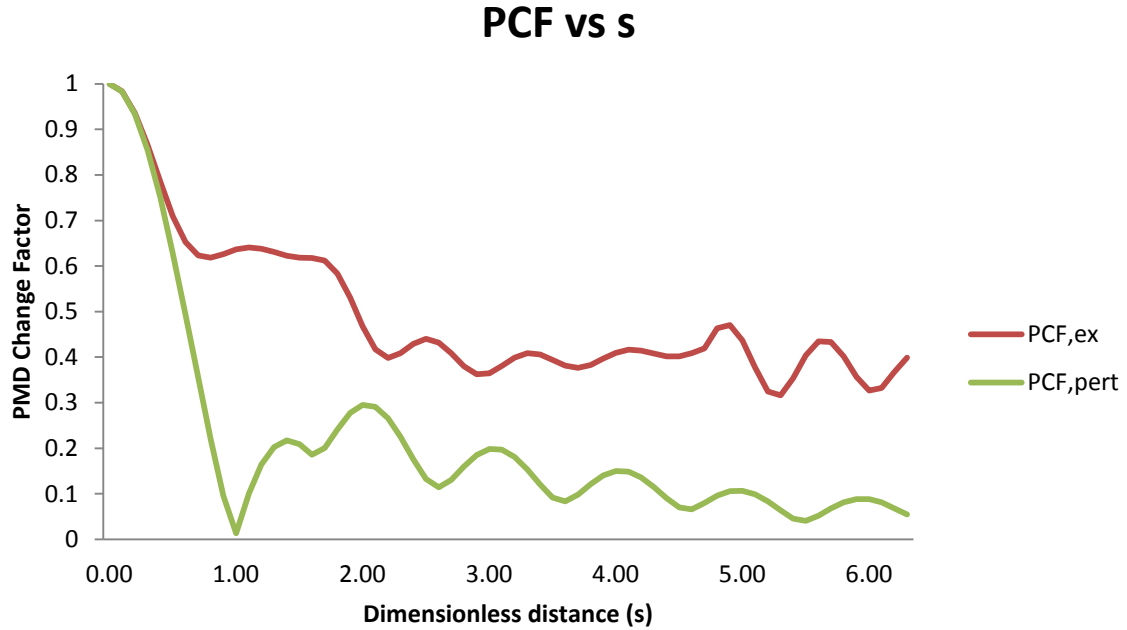


Figure 4. The PCF curve for a nonperturbative limit with $\Lambda = 5$ and $L_B = 1$. The top and bottom curves show exact and perturbative calculations, respectively. Note that perturbative approximation underestimates the PCF in this regime.

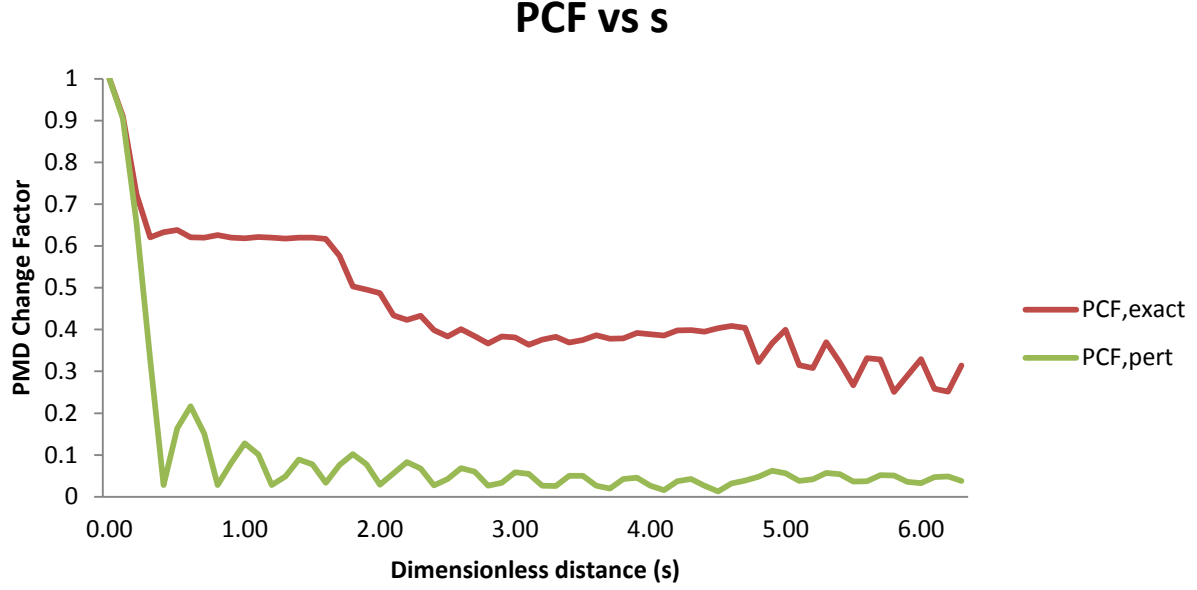


Figure 5. The PCF curve for a nonperturbative limit with $\Lambda = 12$ and $L_B = 1$. The top and bottom curves show exact and perturbative calculations, respectively. It is seen that perturbative approximation underestimates the PCF in this regime.

4. Conclusions

It was shown through these calculations that the perturbative approximation made in Galtarossa et al.² has limited validity compared with an exact calculation. The three-segment approximation given here can be extended to any number of segments. At the next level, the segment expressions can be derived for any given profile of the spin function. The exact analytic expressions allow a physical understanding of the limits of the approximations employed earlier.

² Galtarossa, A.; Palmieri, L.; Pizzinat, A. Optimized Spinning Design for Low PMD Fibers: An Analytical Approach. *J. Lightwave Technol.* **2001**, *19* (10), 1502–1512.

Appendix. Perturbative Calculation for Segments

The perturbative approach is based on the following assumptions:

- The coupling between the polarization states is so small that the equations become decoupled.
- The top component is constant ($A_1^{(m)} = 1, m = 1, 2, 3$), and only the second component changes.
- The boundary conditions remain unchanged.

Under these assumptions, the dimensionless constant q becomes \tilde{c} , which is related to the physical lengths as

$$\tilde{c} = \frac{2}{\pi} \left(\frac{\Lambda}{l_0} \right) \quad (\text{A1})$$

The new equations and their solutions take the following form:

- Segment I: $0 \leq s \leq \frac{\pi}{2}$

Perturbative equations:

$$\begin{bmatrix} A_{1s}^{(1)}(s) \\ A_{2s}^{(1)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\tilde{c}s} \\ e^{-2i\tilde{c}s} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{A2})$$

Solutions:

$$A_2^{(1)}(s) = \left(\frac{a}{\tilde{c}} \right) i e^{-i\tilde{c}s} \sin \tilde{c}s \quad (\text{A3})$$

The sum of squares of the ω -differentiated amplitudes:

$$\left(\frac{|A_{1\omega}^{(1)}(s)|^2 + |A_{2\omega}^{(1)}(s)|^2}{(a_\omega / \tilde{c})^2} \right)_{pert} = \frac{1}{2} (1 - \cos 2\tilde{c}s) = \sin^2 \tilde{c}s \quad (\text{A4})$$

So

$$PCF^{(1)}(s)_{pert} = \left[\frac{\left(|A_{1\omega}^{(1)}(s)|^2 + |A_{2\omega}^{(1)}(s)|^2 \right)_{pert}}{\left[|A_{1\omega}(s)|^2 + |A_{2\omega}(s)|^2 \right]_{unspun-fiber}} \right]^{1/2} = \frac{\sin \tilde{c}s}{\tilde{c}s} \quad (\text{A5})$$

- Segment II: $\frac{\pi}{2} \leq s \leq \frac{3\pi}{2}$

Perturbative equations:

$$\begin{bmatrix} A_{1s}^{(2)}(s) \\ A_{2s}^{(2)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i(-\tilde{c}s+2c)} \\ e^{-2i(-\tilde{c}s+2c)} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{A6})$$

Solutions:

$$A_2^{(2)}(s) = e^{i(\tilde{c}s-2c)} \left(\frac{a}{\tilde{c}} \right) \left[-(1 - \cos 2c) \cos \tilde{c}s + (\sin 2c + i) \sin \tilde{c}s \right] \quad (\text{A7})$$

The sum of squares of the ω -differentiated amplitudes:

$$\left(\left| A_{1\omega}^{(2)}(s) \right|^2 + \left| A_{2\omega}^{(2)}(s) \right|^2 \right)_{pert} = \frac{1}{2} \left(\frac{a_\omega}{\tilde{c}} \right)^2 \left\{ (3 - 2 \cos 2c) + (\cos 4c - 2 \cos 2c) \cos 2\tilde{c}s \right. \\ \left. + (\sin 4c - 2 \sin 2c) \sin 2\tilde{c}s \right\} \quad (\text{A8})$$

Expression for PCF is obtained as before.

- Segment III: $\frac{3\pi}{2} \leq s \leq 2\pi$

Perturbative equations:

$$\begin{bmatrix} A_{1s}^{(3)}(s) \\ A_{2s}^{(3)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i(\tilde{c}s-4c)} \\ e^{-2i(\tilde{c}s-4c)} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{A9})$$

Solutions:

$$A_2^{(3)} = e^{i(-\tilde{c}s+4c)} \left(\frac{a}{\tilde{c}} \right) \left[\begin{aligned} & \left\{ (-1 + \cos 2c + \cos 4c - \cos 6c) \right\} \cos \tilde{c}s \\ & + i \left\{ (-\sin 2c - \sin 4c + \sin 6c) \right\} \cos \tilde{c}s \\ & + \left\{ (\sin 2c + \sin 4c - \sin 6c) \right\} \sin \tilde{c}s \\ & + i \left\{ (1 + \cos 2c + \cos 4c - \cos 6c) \right\} \sin \tilde{c}s \end{aligned} \right] \quad (\text{A10})$$

The sum of squares of the ω -differentiated amplitudes:

$$\left(\left| A_{1\omega}^{(3)}(s) \right|^2 + \left| A_{2\omega}^{(3)}(s) \right|^2 \right)_{pert} = \frac{1}{2} \left(\frac{a_\omega}{\tilde{c}} \right)^2 \left\{ (5 - 4 \cos 4c) \right. \\ \left. + (2 \cos 10c - \cos 8c - 2 \cos 6c) \cos 2\tilde{c}s \right. \\ \left. + (2 \sin 10c - \sin 8c - 2 \sin 6c) \sin 2\tilde{c}s \right\} \quad (\text{A11})$$

The PCF can be calculated as before.

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